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ON EMBEDDING INFINITE CYCLIC COVERS IN COMPACT 3-MANIFOLDS

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1. By a knot we shall mean a smooth knot K in S^3. The exterior of K is $E(K) = S^3 - \operatorname{int} N(K)$, where $N(K)$ is a tubular neighborhood of K, and we shall refer to the infinite cyclic cover of $E(K)$ as simply the infinite cyclic cover of K.

In [JNWZ] the authors consider the question (attributed to John Stallings) of when the infinite cyclic cover of a knot embeds in S^3. Following [JNWZ], we say that such a knot has Property IE. Clearly fibered knots have Property IE. In [JNWZ] it is shown that if a non-fibered genus 1 knot has Property IE then its Alexander polynomial is either 1 or $2 - 5t + 2t^2$, and that in each case there are infinitely many genus 1 knots having Property IE.

[JNWZ] also raises the question of whether the infinite cyclic cover of a knot always embeds in some compact 3-manifold. The following theorem provides a negative answer.

Theorem. *The infinite cyclic cover of the untwisted Whitehead double of a non-trivial knot does not embed in any compact* 3*-manifold.*

Corollary. *There are infinitely many genus* 1 *knots with Alexander polynomial* 1 *whose infinite cyclic covers do not embed in any compact* 3*-manifold.*

2. Let $L = J \cup J'$ be the Whitehead link, and W be its exterior; see Figure 1. Let T be the boundary component of W corresponding to the component J of L, and let m, ℓ be a meridian-longitude pair for J on T. Let k be a non-trivial knot, with exterior X, and let μ, λ be a meridian-longitude pair for k on ∂X. Let K be the untwisted Whitehead double of k. Then the exterior of K is $Y = W \cup_T X$, where T is identified with ∂X so that $m \leftrightarrow \lambda$, $\ell \leftrightarrow \mu$.

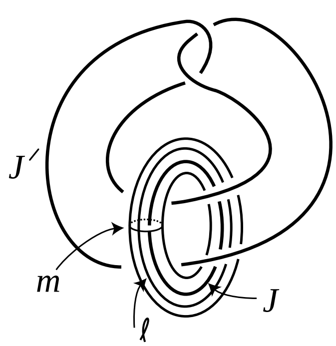

Figure 1

The exterior of J (resp. J') in S^3 is a solid torus V (resp. V'). Let V'_∞ be the infinite cyclic cover of V', and let $\tilde{J}$ be the inverse image of J in V'_∞. The infinite cyclic cover of J' in the solid torus V is $W_\infty = V'_\infty - \operatorname{int} N(\tilde{J})$. Since there is an isotopy of S^3 interchanging

the components of L, the link $\tilde{J}$ in $V'_\infty \cong D^2 \times \mathbb{R}$ is as shown in Figure 2.

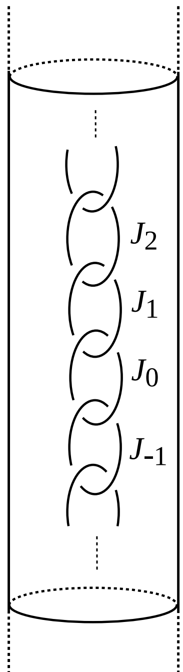

Figure 2

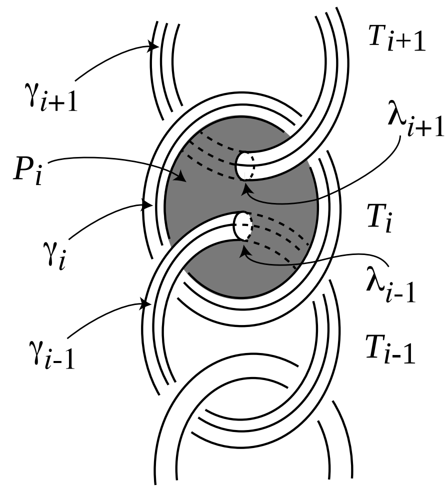

Figure 3

Let the components of $\tilde{J}$ be J_i, $i \in \mathbb{Z}$, indexed in the obvious way, and let $T_i = \partial N(J_i)$. Then the infinite cyclic cover of Y is $Y_\infty = W_\infty \cup (\coprod_{i\in\mathbb{Z}} X_i)$, where X_i is a copy of X, attached to W_∞ by identifying T_i with ∂X_i via $\tilde{m}_i \leftrightarrow \lambda_i$, $\tilde{\ell}_i \leftrightarrow \mu_i$, where μ_i, λ_i is a meridian-longitude pair on ∂X_i, and $\tilde{m}_i, \tilde{\ell}_i$ are lifts of m, ℓ. Note that $\tilde{m}_i$ is a meridian of J_i.

Let $\gamma_i \subset T_i$ be a longitude of J_i in $D^2 \times \mathbb{R}$. Then there is a planar surface $P_i \subset W_\infty$ with $\partial P_i = \gamma_i \cup \lambda_{i-1} \cup \lambda_{i+1}$. See Figure 3.

Let $S_j \subset X_j$ be a copy of a Seifert surface for k, and define $F_i = P_i \cup S_{i-1} \cup S_{i+1}$. Then F_i is an orientable surface in Y_∞ with $F_i \cap X_i = \partial F_i = \gamma_i$. Note that each of γ_{i-1} and γ_{i+1} intersects F_i transversely in a single point.

We are now ready to prove the theorem. So suppose that Y_∞ embeds in a compact 3-manifold M. By passing to a double cover if necessary we may assume that M is orientable.

Lemma. *T_i is incompressible in M.*

Proof. Let D be a compressing disk for T_i in M. Since k is non-trivial, $D \cap X_i = \partial D$. Recall the orientable surface $F_i \subset Y_\infty$ with $\partial F_i \cap X_i = \partial F_i = \gamma_i$. Then ∂D and γ_i are both null-homologous in $\overline{M - X_i}$, and hence they are isotopic on T_i. Let $\widehat{F_i}$ be the (singular) closed surface $F_i \cup D$. We may assume that D intersects T_{i+1} transversely in a finite number of simple closed curves, each essential on T_{i+1}.

If $D \cap T_{i+1} = \emptyset$, then γ_{i+1} intersects $\widehat{F_i}$ transversely in a single point, and so $[\gamma_{i+1}]$ has infinite order in $H_1(M)$. But this contradicts the fact that γ_{i+1} bounds F_{i+1}.

If $D \cap T_{i+1} \neq \emptyset$, then an innermost disk on D is a compressing disk D' for T_{i+1} disjoint from T_i. Now, γ_i meets $\widehat{F}_{i+1} = F_{i+1} \cup D'$ transversely in a single point, and we get a contradiction as before. $\square$

Proof of Theorem. By the Lemma and Haken Finiteness there exist distinct i, j, k such that T_i, T_j and T_k are mutually parallel in M. By relabeling if necessary we may assume that T_j is contained in the interior of the product region between T_i and T_k. Since T_j bounds X_j, either T_i or T_k is contained in X_j, which is absurd. $\square$

References

[JNWZ] B. Jiang, Y. Ni, S. Wang and Q. Zhou, *Embedding infinite cyclic covers of knot spaces into 3-space*, Topology **45** (2006), 691–705.

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